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A theoretical and experimental investigation of the phase configuration of internal waves of small amplitude in a density stratified liquid

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Experiments were conducted to test the linear theory of internal gravity waves produced in a stably stratified liquid by the forced oscillations and the initial impulsive motion of a two-dimensional stationary disturbance. The measurements of the wave configuration in a medium whose density increased linearly with depth were made by means of a Toepler-schlieren system. The agreement between observation and prediction was found to be good.

1. Introduction

The generation of internal gravity waves by a small disturbance in a fluid of non-uniform density or entropy was first discussed by Love and his results were repeated by Lamb (1932); see §2 below. Yih (1965) discussed small amplitude internal waves in some detail, but was concerned primarily with the theoretical problems associated with waves in bounded media. To the authors' knowledge, the simple properties of the phase configuration of internal waves had not been investigated experimentally by techniques which permitted a precise comparison of theory and experiment to be made nor had many of the implications of the theory been verified. Görtler (1943) discussed the properties of the steady-state wave system in terms of the characteristics of the linearized equations of motion and performed experiments using a shadowgraph technique. This work came to the authors' notice some months after the present investigation was completed. The present paper differs for Görtler's in that the analysis is couched in terms of group velocity arguments, which is perhaps more in the spirit of a linearized theory. The paper goes beyond that of Görtler in discussing propagation in a medium inhomogeneous with respect to the waves, in deriving the law of reflexion, in examining the trapping of waves and in analysing the transient problem. The steady-state wave configuration in a medium homogeneous with respect to the waves was studied and essentially the same experimental results obtained using a similar technique by Görtler.

The paper gives an account of an investigation into two simple situations, in which small amplitude waves are produced by a small disturbance which is localized near the origin of space co-ordinates, and which either oscillates with a definite frequency ω or is moved impulsively to simulate a point disturbance in

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time. The general relation between frequency and wave-number is rederived from which the group velocity is deduced using stationary phase arguments. It is shown that, for wavelengths small compared with the distance over which the density changes appreciably, the group and phase velocities are at right angles to one another, and that, in the steady state, there is a single direction depending on frequency in which the waves may propagate. The Cauchy–Poisson problem is solved to give the form of the wave-crests and disturbance front for an impulsive disturbance; we find that there are two families of waves, of similar appearance to the Kelvin ship-waves, although the disturbance is stationary in the present problem.

Experiments were performed in which such waves were generated by a circular cylinder oscillating horizontally with small amplitude or a flat strip with its face vertical moved abruptly through a small distance in a horizontal direction in a tank carefully filled with a salt-water solution whose salinity varied with depth and for which the frequency $(-g\rho^{-1}d\rho/dy)^{\frac{1}{2}}$ was nearly constant; the significance of this parameter will become clear later. It has recently been realized that it is possible to observe and to take precise measurements of the phase configuration of internal waves by means of a schlieren system similar to that used to observe flow fields in which the density is non-uniform by reason of compressibility; see Mowbray (1966). In the present situation, the effects of compressibility are completely unimportant; the density is non-uniform initially and subsequent disturbances produce deviations from the original non-uniformity. It is these which are measured. The agreement between the experimental measurements and the predictions of the wave-crest pattern by the theory was good; no attempt has been made to discuss the amplitude variation. The main features of the theory were observed and no phenomenon was observed which was not readily explicable in terms of the theory.

We begin by giving a brief outline of the theory, noting several implications which can be tested. We then outline the experimental arrangements and techniques and discuss the results, concluding with the comparisons between observation and prediction. The agreement was found to be better than 5 %.

2. The frequency wave-number relation and its implications

If ρ is the density, $\mathbf{q}$ the velocity vector, p the pressure and $\mathbf{g}$ the acceleration due to gravity, then in a co-ordinate system with the x-axis horizontal and the y-axis vertical with y increasing upwards, the equations of motion are

$$(\partial\rho/\partial t)+\nabla.(\rho\mathbf{q}) = 0, \tag{1}$$

$$\rho(\partial\mathbf{q}/\partial t)+\rho\mathbf{q}.\nabla\mathbf{q} = -\nabla p+\rho\mathbf{g}, \tag{2}$$

together with the condition that the density is constant along a particle path

$$(\partial\rho/\partial t)+\mathbf{q}.\nabla\rho = 0. \tag{3}$$

Equations (1) and (3) imply that

$$\nabla.\mathbf{q} = 0. \tag{4}$$

If we denote equilibrium values of quantities in the medium at rest by a suffix zero and disturbances from equilibrium by a suffix one, so that $\rho = \rho_0+\rho_1$, and if

$\mathbf{q} = (u, v)$, then the equations governing the propagation of small disturbances are

$$\rho_0(\partial u/\partial t) = -\partial p_1/\partial x,$$
$$\rho_0(\partial v/\partial t) = -(\partial p_1/\partial y) - \rho_1 g,$$
$$(\partial \rho_1/\partial t) + v(d\rho_0/dy) = 0,$$

and
$$(\partial u/\partial x) + (\partial v/\partial y) = 0,$$

where we have used the equilibrium condition

$$\partial p_0/\partial y = -\rho_0 g.$$

We define ψ by the relations

$$u = \partial\psi/\partial y, \quad v = -\partial\psi/\partial x.$$

Then, substituting for u and v, eliminating the pressure gradient terms, we have

$$\frac{d\rho_0}{dy}\frac{\partial^2\psi}{\partial t\,\partial y} + \rho_0\left(\frac{\partial^3\psi}{\partial t\,\partial y^2} + \frac{\partial^3\psi}{\partial t\,\partial x^2}\right) = g\frac{\partial \rho_1}{\partial x}.$$

Eliminating derivatives of ρ_1, we get

$$\nabla^2\ddot{\psi} - \frac{\omega_0^2}{g}\frac{\partial\ddot{\psi}}{\partial y} + \omega_0^2\frac{\partial^2\psi}{\partial x^2} = 0,$$

where a dot denotes differentiation with respect to t and $\omega_0^2 = -g\rho_0^{-1}d\rho_0/dy$ is the square of the local Väisälä–Brunt frequency. This equation was derived by Love and repeated by Lamb (1932).

If we look for a solution of the form

$$\psi = \psi_0(y)\exp\{i(k_1x + k_2y - \omega t)\},$$

then
$$\omega^2 = \omega_0^2k_1^2(k_1^2 + k_2^2 - 2ik_2\psi_0'/\psi_0 - \psi_0''/\psi_0 + ik_2\omega_0^2/g + \omega_0^2\psi_0'/g\psi_0)^{-1},$$

where a dash denotes differentiation with respect to y. The medium is defined to be *homogeneous with respect to internal gravity waves* if ω_0 is constant throughout it. In such a medium, conservation of energy requires that ψ_0 must be proportional to $\exp(\omega_0^2y/2g)$. Thus

$$\omega^2 = \omega_0^2k_1^2(k_1^2 + k_2^2 + (\omega_0^2/2g)^2)^{-1}, \tag{5}$$

and
$$\psi \propto \exp(\omega_0^2y/2g)\exp\{i(k_1x + k_2y - \omega t)\}.$$

If $(\omega_0^2/2g)^2 \ll k_1^2 + k_2^2 = 4\pi^2/\lambda^2$, where λ is the wavelength of the waves, then

$$\omega = \omega_0\sin\theta\{1 - \tfrac{1}{8}\pi^{-2}\lambda^2(\omega_0^2/2g)^2 + \ldots\},$$

where $\sin\theta = \pm k_1(k_1^2 + k_2^2)^{-\frac{1}{2}}$, so that θ is the angle between the wave-number vector $\mathbf{k}$ and the vertical. Note that the relation $\omega = \omega_0\sin\theta$ does not depend on the magnitude of $\mathbf{k}$ but only on its direction, so that for a given frequency ω and a given value of ω_0, waves of all wavelengths may be excited.

It is well known that in linear one-dimensional wave systems the energy travels with the group velocity $c_g = d\omega/dk$. For two-dimensional disturbances, it can similarly be shown that energy travels with the group velocity

$$\mathbf{c}_g = (\partial\omega/\partial k_1,\ \partial\omega/\partial k_2).$$

If we consider a wave train which has been produced by a disturbance of definite frequency which has been operating for all time, the waves may be represented by a Fourier integral of the form

$$\psi = \int f(\mathbf{k}) \exp\{i(k_1 x + k_2 y - \omega t) d\mathbf{k}, \tag{6}$$

and sufficiently far from the disturbing mechanism, at large x, say, we may deduce the asymptotic value of the integral (6) by the principle of stationary phase. This states that the most important contribution to the integral

$$\int f(\mathbf{k}) \exp\{ig(\mathbf{k}, \omega, y, t)\, x\}\, d\mathbf{k}$$

arises from points at which $\partial g/\partial k_1 = \partial g/\partial k_2 = 0$, provided these stationary points are distinct; the contribution to the integral is $O(x^{-1})$. Here, $g = k_1 + k_2 y/x - \omega t/x$, so that the conditions for points of stationary phase are

$$1 - \frac{\partial \omega}{\partial k_1} \frac{t}{x} = 0$$

and

$$\frac{y}{x} - \frac{\partial \omega}{\partial k_2} \frac{t}{x} = 0,$$

which we may write as

$$\frac{y}{x} = \frac{\partial \omega}{\partial k_2} \bigg/ \frac{\partial \omega}{\partial k_1}.$$

These results can also be deduced from consideration of the kinematics of wave crests; see, for example, Whitham (1960). Alternatively, we may write the equations in the form

$$x/t = \partial \omega/\partial k_1 \quad \text{and} \quad y/t = \partial \omega/\partial k_2 \tag{7}$$

with $x/y = O(1)$. This may be interpreted as stating that the main contribution to the integral propagates with the velocity $(\partial \omega/\partial k_1, \partial \omega/\partial k_2)$. If the points of stationary phase are not distinct, so that

$$\frac{\partial^2 g}{\partial k_1^2} \frac{\partial^2 g}{\partial k_2^2} - \left(\frac{\partial^2 g}{\partial k_1 \partial k_2}\right)^2 = 0, \tag{8}$$

in addition to (7), then the main contribution is $O(x^{-\frac{5}{6}})$; see, for example, Jones & Kline (1958). This is the argument which establishes the envelope of the Kelvin ship-wave pattern, for example.

We may consider the waves produced by an initial disturbance localized in space and time. The waves may again be represented by a Fourier integral, but over $\mathbf{k}$ and ω, and after a sufficiently long time, the main contribution to the integral considered in the form $\int f(\mathbf{k}) \exp\{ih(x, y, \mathbf{k}, \omega)\, t\} d\mathbf{k}\, d\omega$ arises from points at which $\partial h/\partial k_1 = \partial h/\partial k_2 = 0$ provided these are distinct. Here,

$$h = k_1 x/t + k_2 y/t - \omega,$$

so that the conditions for points of stationary phase are again (7); the contribution to the integral is $O(t^{-1})$. If the points of stationary phase are not distinct, the contribution is $O(t^{-\frac{5}{6}})$. This form of the argument allows the form of the disturbance front which propagates outwards from an initial disturbance to be derived. In §4, in which we consider the Cauchy–Poisson problem for internal waves, the analysis reveals the properties of steady-state waves without recourse to arguments of stationary phase. For a much fuller discussion of group velocity and methods of stationary phase, the reader is referred to Lighthill (1965).

For the remainder of this section we shall consider the phase properties of short steady-state waves using group velocity concepts; see also Eckart (1960). For short waves, the dispersion relation reduces to $\omega = \omega_0 k_1(k_1^2+k_2^2)^{-\frac{1}{2}}$ and the group velocity is $\mathbf{c}_g = \omega_0 k_2(k_1^2+k_2^2)^{-\frac{3}{2}}(k_2, -k_1)$, so that $\mathbf{c}_g$ is perpendicular to $\mathbf{k}$, that is, the waves possess the property of propagating outwards along their crests. This follows since the phase velocity of a wave is $\omega\mathbf{k}/|k|^2$ so that crests and troughs appear to travel in the direction of $\mathbf{k}$, that is they are themselves perpendicular to $\mathbf{k}$. But waves can manifest themselves only in the direction in which energy is propagated, that is in the direction of $\mathbf{c}_g$ which is parallel to the crests and troughs. Further, we see that associated with a given frequency ω, which must be less than the Väisälä–Brunt frequency ω_0, there is a unique direction of the wave-number vector $\mathbf{k}$, the angle θ to the vertical. There is therefore a unique direction of $\mathbf{c}_g$, the angle θ to the horizontal. Since we are here concerned with point disturbances in the (x, y)-plane, we arrive at the conclusion that steady-state waves from a point disturbance of frequency ω will be observed as a cross with its branches at the angle $\sin^{-1}(\omega/\omega_0)$ to the horizontal. If $\omega \ll \omega_0$, the crests will be almost horizontal; if $\omega = \omega_0$, the crests will be vertical; if $\omega > \omega_0$, no small amplitude waves are permissible.

A deduction from this result is that, in a practical situation in which a disturbance has associated with it a fixed frequency, but is not ideally sinusoidal, one would expect some energy to be vested in the harmonics. However, only harmonics whose frequency is less than ω_0 will appear. A further deduction is that the velocity disturbances are everywhere perpendicular to $\mathbf{k}$, that is the waves are transverse waves. Hence, the velocity disturbances in a strictly two-dimensional situation are parallel to $\mathbf{c}_g$ so that particles are constrained to oscillate along the group velocity vector and there is a non-zero velocity parallel to the wave crests. Direct observation of the path of a particle engaged in this wave motion would suggest that the wave was purely longitudinal, which is completely mistaken.

Another prediction is that of the possibility of the trapping of waves by allowing the medium to be suitably inhomogeneous so that ω_0 is not constant. Consider, for example, waves generated by a small disturbance of frequency ω operating in a region of constant ω_0 and starting at time $t = 0$. The resulting waves would be seen to propagate outwards with their crests along the direction θ to the horizontal; in practice, the waves to be expected from the discontinuity of the starting process are weak. For practical purposes, the emergence of a strong wave from the inner region surrounding the oscillating bar marks the beginning of the 'steady-state' regime. If, however, after some distance the wave encounters a region of slowly varying ω_0, then since ω is fixed along the ray, the direction of the ray, θ to the horizontal, must change to compensate. If we introduce a phase function ϕ with the properties that $\omega = -\partial\phi/\partial t$ and $\mathbf{k} = \nabla\phi$, then differentiating the dispersion relation $\omega = f(\mathbf{k}, \mathbf{r})$ with respect to a space co-ordinate and re-interpreting the derivatives, we obtain successively

$$\frac{\partial k_1}{\partial t} + \frac{\partial f}{\partial k_1}\frac{\partial k_1}{\partial x} + \frac{\partial f}{\partial k_2}\frac{\partial k_1}{\partial y} = 0,$$

and

$$\frac{\partial k_2}{\partial t} + \frac{\partial f}{\partial k_1}\frac{\partial k_2}{\partial x} + \frac{\partial f}{\partial k_2}\frac{\partial k_2}{\partial y} = -\frac{\partial f}{\partial y},$$

since ω is independent of x. This shows that k_1 is unchanged along the ray, but that k_2 changes according to the rate at which the ray ascends or descends. Thus, θ changes. For example, if ω_0 increases then the crests are bent away from the vertical and if ω_0 increases sufficiently the crests will ultimately propagate horizontally. If ω_0 decreases, the wave-crests are bent towards the vertical, and if ω_0 decreases to ω, they will become vertical. If ω_0 continues to decrease then small amplitude progressive waves are forbidden, so that waves may not propagate beyond the region in which $\omega_0 = \omega$; the waves have been trapped. We shall discuss this in more detail when we consider the experimental results.

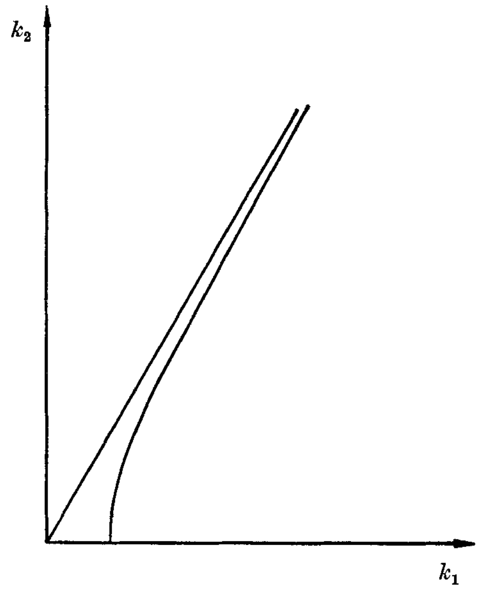

FIGURE 1. The curve ω as a function of $\mathbf{k}$.

3. Steady-state waves of arbitrary wavelength

Let us now consider the case of arbitrary wavelength; although there is little energy vested in long waves in many internal wave systems, it is, nevertheless, important to know how the short wavelength results are to be modified. The curve of constant frequency, see (5) above, is a hyperbola in the $\mathbf{k}$-plane and is shown in figure 1. The k_1-intercept Δ is $\omega(\omega_0^2/2g)(\omega_0^2-\omega^2)^{-\frac{1}{2}}$ and we can see that the asymptotes are the lines $k_1/k_2 = \pm\tan\theta$. Moreover, there is a maximum wavelength $\lambda_{\max} = 2\pi/\Delta$. The permissible directions for group velocity are the normals to the curve $\omega =$ constant, and we can readily see that these lie in a fan between the horizontal and the directions $\pm\theta$ to the horizontal; see figure 2. Of course, whether or not a disturbance is ever observed in a particular direction depends on what fraction of the total energy is associated with the particular $\mathbf{k}$ to which the direction corresponds.

We have pointed out that ordinary group velocity arguments are applicable only when points of stationary phase are simple. The condition for double stationary points (8) requires

$$\{k_1^2+k_2^2+(\omega_0^2/2g)^2\}\{k_2^2(k_1^2+k_2^2+(\omega_0^2/2g)^2)-3k_1^2(\omega_0^2/2g)^2\}=0, \tag{9}$$

which yields a relation between k_1 and k_2. Coupled with the relation for stationary phase, this gives the form of the disturbance front in the Cauchy–Poisson problem which we shall consider in the next section. However, if we insist that the

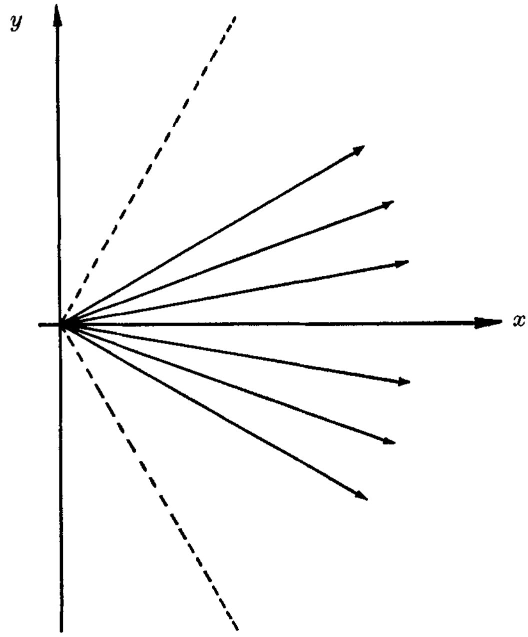

FIGURE 2. The directions of the group velocity vectors for waves of any wavelength.

waves are short, that is $(\omega_0^2/2g)^2 \ll k_1^2+k_2^2$, equation (9) requires $k_2=0$, with k_1 arbitrary. Such waves have their crests vertical and have zero group velocity. For short waves generated with frequency ω which is less than ω_0, this implies that there are no double stationary points. We shall consider the implications for waves generated in a medium inhomogeneous in ω_0 when we discuss the experimental results.

4. The Cauchy–Poisson problem

The equation for the disturbance stream function ψ in the presence of a force $\mathbf{F}=(F_x,F_y)$ is easily shown to be

$$\nabla^2\ddot{\psi}-\frac{\omega_0^2}{g}\frac{\partial\ddot{\psi}}{\partial y}+\omega_0^2\frac{\partial^2\psi}{\partial x^2}=\frac{1}{\rho_0}\left\{\frac{\partial^2}{\partial t\,\partial y}(F_x)-\frac{\partial^2}{\partial t\,\partial x}(F_y)\right\}.$$

If we put $\psi = \Psi \exp(\omega_0^2 y/2g)$, then Ψ satisfies the equation

$$\nabla^2\Psi - \left(\frac{\omega_0^2}{2g}\right)^2 \Psi + \omega_0^2 \frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{\rho_{00}} \exp(\omega_0^2/2g)\left\{\frac{\partial^2}{\partial t\,\partial y}(F_x) - \frac{\partial^2}{\partial t\,\partial x}(F_y)\right\},$$

where $\rho_0 = \rho_{00} \exp(-\omega_0^2 y/g)$. If F_x and F_y are proportional to $\delta(x)\,\delta(y)\,\delta(t)$, where δ denotes the Dirac delta function, we may express Ψ as a linear combination of

$$I = \int_{-\infty}^{\infty} \frac{i\omega \exp\{i(k_1 x + k_2 y - \omega t)\}}{\omega^2(k_1^2 + k_2^2) - \omega_0^2 k_1^2 + (\omega_0^2/2g)^2\omega^2}\, dk_1\, dk_2\, d\omega$$

and its first derivatives with respect to x and y. Putting $\omega = \alpha\omega_0$, we have

$$I = \int_{-\infty}^{\infty} \frac{i\alpha \exp\{i(k_1 x + k_2 y - \alpha\omega_0 t)\}}{k_1^2(\alpha^2-1) + \alpha^2(k_2^2 + (\omega_0^2/2g)^2)}\, dk_1\, dk_2\, d\alpha.$$

We may effect the integration with respect to k_1 by considering

$$\int_C \exp(izx)\{z^2(\alpha^2-1) + B^2\}^{-1} dz$$

round the large semi-circle C in the upper half plane; α and B are real. The poles of the integrand are simple and lie at the points $\pm B(1-\alpha^2)^{-\frac{1}{2}}$. If $\alpha < 1$, these lie on the real axis; the contour must be indented in an asymmetric fashion to ensure that the waves are outgoing. If $\alpha > 1$, the poles are purely imaginary with only one inside the contour. For $\alpha < 1$

$$\int_{-\infty}^{\infty} \frac{\exp\{i(k_1 x + k_2 y)\}}{k_1^2(\alpha^2-1) + \alpha^2(k_2^2 + (\omega_0^2/2g)^2)}\, dk_1\, dk_2$$
$$= -\pi i \int_{-\infty}^{\infty} \frac{\exp[i\{x\alpha/(1-\alpha^2)^{\frac{1}{2}}[k_2^2 + (\omega_0^2/2g)^2]^{\frac{1}{2}} + k_2 y\}]}{(1-\alpha^2)^{\frac{1}{2}}\alpha(k_2^2 + (\omega_0^2/2g)^2)^{\frac{1}{2}}}\, dk_2,$$

where we shall consider $x > 0$. Now consider the integral

$$G_1(r,\theta) = \int_{-\infty}^{\infty} \frac{\exp\{i(r(t^2+1)^{\frac{1}{2}}\cosh\theta + rt\sinh\theta)\}}{(t^2+1)^{\frac{1}{2}}}\, dt.$$

If we let $t = \sinh\phi$, then

$$G_1(r,\theta) = \int_{-\infty}^{\infty} \exp\{ir\cosh(\phi+\theta)\}\, d\phi$$

which is obviously independent of θ, so that in particular

$$G_1(r,\theta) = G_1(r,0) = \pi i H_0^{(1)}(r).$$

Hence,
$$I_{\alpha<1} = \pi^2 \int_{-1}^{1} \frac{\exp(-i\omega_0\alpha t)}{(1-\alpha^2)^{\frac{1}{2}}} H_0^{(1)}\left[\frac{\omega_0^2}{2g}\left(\frac{x^2\alpha^2}{1-\alpha^2} - y^2\right)^{\frac{1}{2}}\right] d\alpha.$$

For $\alpha > 1$

$$\int_{-\infty}^{\infty} \frac{\exp\{i(k_1 x + k_2 y)\}}{k_1^2(\alpha^2-1) + \alpha^2(k_2^2 + (\omega_0^2/2g)^2)}\, dk_1\, dk_2$$
$$= \pi \int_{-\infty}^{\infty} \frac{\exp[-\alpha x/(\alpha^2-1)^{\frac{1}{2}}\{k_2^2 + (\omega_0^2/2g)^2\}^{\frac{1}{2}} + ik_2 y]}{\alpha(\alpha^2-1)^{\frac{1}{2}}(k_2^2 + (\omega_0^2/2g)^2)^{\frac{1}{2}}}\, dk_2.$$

Consider
$$G_2(r,\theta) = \int_{-\infty}^{\infty} \frac{\exp\{-r(t^2+1)^{\frac{1}{2}}\cos\theta + irt\sin\theta\}}{(t^2+1)^{\frac{1}{2}}}\, dt.$$

If we put $t = \sinh\theta$ as before, we may likewise show that

$$G_2(r,\theta) = G_2(r,0) = \pi i H_0^{(1)}(ir).$$

Thus, we may write

$$I = \pi^2 \int_{-\infty}^{\infty} \frac{\exp(-i\omega_0 \alpha t)}{(1-\alpha^2)^{\frac{1}{2}}} H_0^{(1)}\left[\frac{\omega_0^2}{2g}\left(\frac{x^2\alpha^2}{1-\alpha^2} - y^2\right)^{\frac{1}{2}}\right] d\alpha,$$

where the positive root of the argument of $H_0^{(1)}$ is to be taken. Now,

$$(d/dz) H_0^{(1)}(z) = -H_1^{(1)}(z),$$

so that the integral over α which represents Ψ has an integrand of the form

$$\frac{\exp(-i\omega_0\alpha t)}{(1-\alpha^2)^{\frac{1}{2}}}\left\{C H_0^{(1)}\left[\frac{\omega_0^2}{2g}\left(\frac{x^2\alpha^2}{1-\alpha^2} - y^2\right)^{\frac{1}{2}}\right] + D H_1^{(1)}\left[\frac{\omega_0^2}{2g}\left(\frac{x^2\alpha^2}{1-\alpha^2} - y^2\right)^{\frac{1}{2}}\right]\right\},$$

where C and D are non-oscillatory functions of x, y and α.

We note that the steady-state problem has this integrand itself as the representation of Ψ and that it changes its character as the argument of the Bessel functions change from a real to an imaginary quantity; this occurs at

$$x^2\alpha^2(1-\alpha^2)^{-1} = y^2.$$

This is the line $\alpha = \sin\theta$, or $\omega = \omega_0 \sin\theta$ where $y/x = \tan\theta$.

To return to the Cauchy–Poisson problem: the asymptotic form of the Bessel functions is given by

$$H_\nu^{(1)}(z) \sim (2/\pi z)^{\frac{1}{2}} \exp\{i(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\}\, {}_2F_0(\tfrac{1}{2}+\nu,\ \tfrac{1}{2}-\nu;\ 1/2iz).$$

Thus, Ψ is proportional to

$$\int_{-\infty}^{\infty} f\left(\alpha, \frac{x}{t}, \frac{y}{t}\right) \exp\left[i\left\{\frac{\omega_0^2}{2g}\left(\frac{\alpha^2}{1-\alpha^2}\left(\frac{x}{t}\right)^2 - \left(\frac{y}{t}\right)^2\right)^{\frac{1}{2}} - \omega_0\alpha\right\} t\right] d\alpha,$$

where t is to be the large parameter and the function f is non-oscillatory. The points of stationary phase are

$$\frac{\omega_0^2}{2g}\frac{d}{d\alpha}\left[\frac{\alpha^2}{1-\alpha^2}\left(\frac{x}{t}\right)^2 - \left(\frac{y}{t}\right)^2\right]^{\frac{1}{2}} = \omega_0,$$

that is

$$\left(\frac{\alpha^2}{1-\alpha^2}\xi^2 - \eta^2\right)^{\frac{1}{2}} = \frac{\alpha}{(1-\alpha^2)^2}\xi^2,$$

where $\xi = (\omega_0/2g)(x/t)$ and $\eta = (\omega_0/2g)(y/t)$. Crests and troughs are the loci of points of constant phase

$$\left(\frac{\alpha^2}{1-\alpha^2}\xi^2 - \eta^2\right)^{\frac{1}{2}} = \alpha + \beta/t,$$

where β is a parameter such that one moves from crest to crest by changing β by $2\pi/\omega_0$. The locus of double stationary points gives the form of the disturbance front; this has the parametric equation

$$\xi = (1-\alpha^2)^{\frac{3}{2}}(1+3\alpha^2)^{-\frac{1}{2}},$$
$$\eta = \sqrt{3}\,\alpha^2(1-\alpha^2)(1+3\alpha^2)^{-1} \quad (0 \leqslant \alpha \leqslant 1),$$

and is shown in figure 3 as a dashed curve. A crest or trough is given by the parametric relation

$$\xi = \alpha^{-\frac{1}{2}}(\alpha+\beta/t)^{\frac{1}{2}}(1-\alpha^2),$$
$$\eta = (\alpha+\beta/t)^{\frac{1}{2}}(-\beta/t-\alpha^3)^{\frac{1}{2}},$$

where $\alpha\beta < 0$ and $|\beta/t| \leqslant |\alpha| \leqslant |\beta/t|^{\frac{1}{3}} \leqslant 1$, for fixed β. In figure 3, the full curves are curves of constant phase for given values of β/t. We see that there are two families of waves which we call the oblique and transverse waves by analogy with the Kelvin ship-wave pattern. The intersection of a particular curve with the

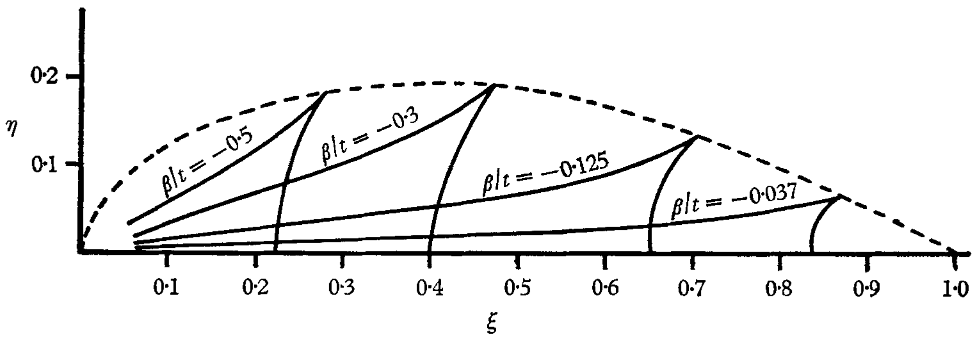

FIGURE 3. The wave envelope and curves of constant β/t.

envelope is a cusp and corresponds to $4\alpha^3 + 3(\beta/t)\,\alpha^2 + \beta/t = 0$. Since neighbouring crests have values of β which differ by $2\pi/\omega_0$, these full curves do not correspond to crests at all instants of time. Since $|\beta/t| \leqslant 1$, the number of crests increases linearly with time, and in such a way that an observer moving with constant velocity always observes waves of fixed wavelength. This follows from the fact that, having specified the group velocity components, the wavelength is determined. On figure 3, the observer would be positioned at a fixed point. Crests would move in such a manner that the oblique waves would move towards the ξ-axis, new waves being created in the upper left hand part of the lobe. The result would be an increasing number of full curves in the lobe, more and more closely spaced as time progressed, the spacing decreasing at precisely the rate required to preserve constant wavelength at a point in the diagram, compensating for a scale of distance which shrinks with time. We have chosen four arbitrary values of β/t for the sake of illustration.

Let us now return to the case of short steady-state waves. We describe the experimental arrangements and show that the waves observed are indeed short.

5. The experimental investigations

These were carried out in a glass-walled rectangular tank, 50 cm square by 100 cm deep, containing a salt solution with density increasing linearly from top to bottom. The methods used for obtaining such a density stratified medium have been discussed in detail by Mowbray (1966), in which paper the Toepler–schlieren technique used in the experiments is also described. The distance characteristic of appreciable density variations, $(\rho^{-1}d\rho/dy)^{-1}$, was typically

500 cm. The disturbance had a typical diameter of about 2 cm, producing waves of similar wavelength, so that the observed waves were indeed short.

The waves were observed by a schlieren technique which is sensitive to variations in refractive index; there is assumed to be a one to one correspondence between refractive index and density. It is important to note that a schlieren

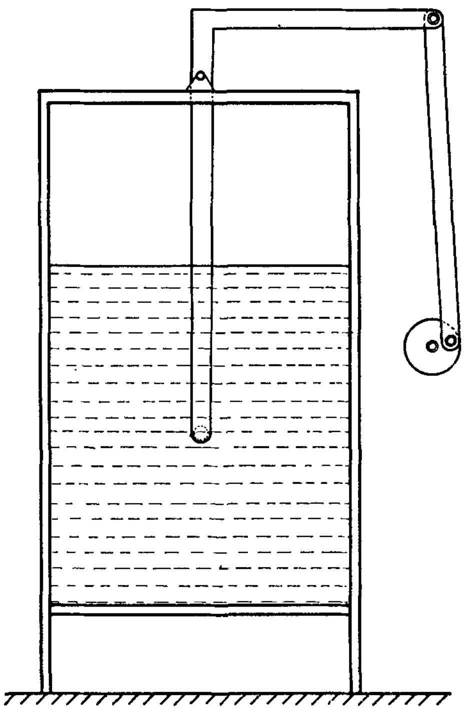

FIGURE 4. A sketch of the apparatus.

system is usually employed to detect departures from a uniform distribution of refractive index, corresponding to the undisturbed state of the medium. Here, the undisturbed state has itself a variation of refractive index, and the schlieren system must be set up to accommodate this before measurements are taken. As the emergent light beam from the undisturbed medium must consist of parallel rays, the density gradient in the tank must be constant so that the incident parallel beam remains parallel after its passage through the medium; see Mowbray (1966). This implies that ω_0 is of the form $(b+cy)^{-\frac{1}{2}}$ which is not constant, so that the medium is not homogeneous with respect to ω_0. However, the departures from homogeneity are small and result in the slight bending of an otherwise straight crest. As far as the phase properties of the wave system are concerned, this small inhomogeneity is unimportant, and is actually useful in that it allows one to correlate the bending of the crest and the change in ω_0.

At time $t = 0$, the waves were generated by the forced oscillations of a bar of about 2 cm diameter, supported horizontally midway in the tank by thin struts of elliptic section hinged at their upper end (see figure 4), with a frequency which was a fraction of the Väisälä–Brunt frequency ω_0; a typical value of ω_0 is $0{\cdot}5\ \mathrm{sec}^{-1}$ and this represents the maximum permissible frequency.

Plate 1 (1) shows the image of the fluid in the absence of waves; as in all the following plates, the knife-edge is vertical. The horizontal band arises as a result of a phenomenon which is peculiar to anisotropic fluids. To discuss it we must digress briefly.

A particle in a stratified fluid of infinite extent in which a horizontal cylinder of radius a moves in a horizontal direction is acted on by the pressure field of the cylinder and the vertical restoring force due to its own buoyancy. Let us measure the displacement y^* from the horizontal plane through the centre of the cylinder and let x^* be the distance from the centre of the cylinder in the horizontal direction. If the cylinder is moving sufficiently slowly, we can find a $\beta < 1$ for which a particle, originating in the layer $y^* = \beta a$, has exactly balancing vertical pressure gradient and buoyancy force at that station in its path at which $y^* = a$; particles from layers $y^* < \beta a$ have balancing pressure gradient and buoyancy forces at $y^* < a$. Since the pressure gradient decreases with increasing x^*, say, β^* will not be uniform in x^*, but will be uniform in time. This implies that all particles originating in layers $y^* < \beta \alpha$ can never flow past the cylinder. The medium is incompressible so that, to satisfy the continuity equation, the cylinder must push ahead and drag behind a slab of fluid, stationary with respect to itself in the steady case, which has a width equal to its own diameter. Note that this phenomenon exists in an inviscid fluid. The far field can be considered to be a wave of zero frequency, that is $k_1 = 0$ with k_2 arbitrary. The group velocity has components $(\omega_0/k_2, 0)$ and is wholly in the horizontal direction and is finite; likewise the particle velocity $(\partial\psi/\partial y, -\partial\psi/\partial x)$ is wholly in the horizontal direction. That is, there is a horizontal column being pushed ahead of and dragged behind the cylinder. The phase velocity is zero, since it is proportional to frequency. The wavelength is $2\pi U/\omega_0$ for the crests to be stationary with respect to the cylinder, so that the rate at which the head of the column advances, that is the group velocity, is the velocity of the cylinder. The authors understand that Dr Bretherton has made a detailed study of this problem.

In a bounded medium, a more complicated but similar phenomenon occurs; β is no longer uniform in time. Those particles, which in the fluid of infinite extent are pushed ahead, say, are unable to move unhindered in the x^*-direction; the slab is 'squeezed' between the cylinder and the bounding wall, and thickens. Fluid 'spills' over the cylinder, driven by its own buoyancy force which is enhanced because of the corresponding reduction in thickness of the layer dragged behind the cylinder, 'stretched' between the cylinder and the wall. One would expect some viscous mixing to take place near the top and bottom edges of the slab, creating a thin layer of more or less constant density, which appears as a discontinuity of contrast in the schlieren photograph. Some considerable time must elapse before this non-uniformity of density gradient disappears under the action of diffusion. The horizontal band in plate 1 (1), is therefore a legacy from previous runs, and its effect is confined to its immediate neighbourhood.

The tank is filled from the bottom by running in layers of successively greater density, and, despite efforts made to keep the tank clean, the water used in the experiments contained a small amount of very fine 'silt' which gradually settled; fluid directly under the bar is shielded by the bar and contains less silt than the surrounding fluid, so that it appears as a light region on the photograph. This phenomenon has no effect on the wave system.

Plate 1 (2), shows waves corresponding to $\omega = 0{\cdot}318\omega_0$. We see that they are close to the horizontal, and, as we would expect, appear both above and below the horizontal line through the initial disturbance. Note that there are no disturbances visible elsewhere in the field of view. First harmonics are present but do not contain enough energy to manifest themselves.

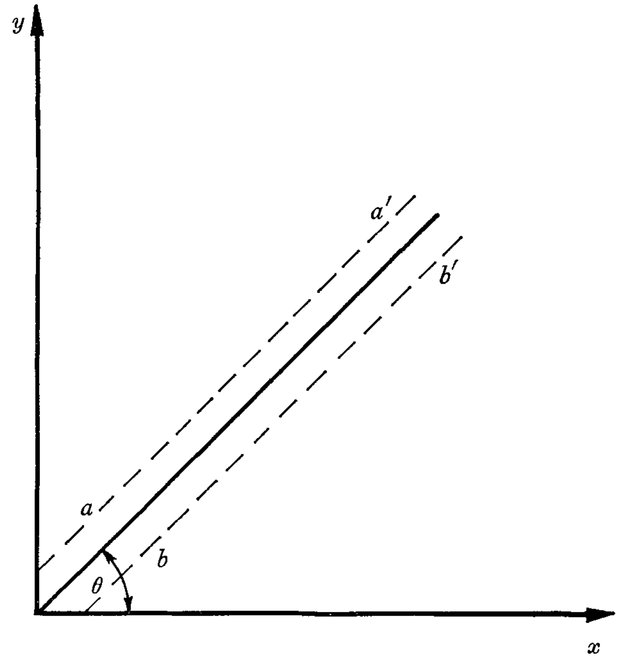

FIGURE 5. A diagram of successive wave crest positions.

Plate 1, figure 3, shows waves at $\omega = 0{\cdot}366\omega_0$. Examination of the wave motion, and of ciné film of the wave motion, shows quite clearly that waves do propagate along their crests and troughs and that their phase and group velocities are perpendicular. The disturbances are, as one would expect, not ideally thin lines, so that a wave crest appears to be created at the line *aa'* (see figure 5) and moves across to the line *bb'* and is then annihilated, the crest simultaneously lengthening. It is important to note that since $\omega = \omega_0 k_1(k_1^2+k_2^2)^{-\frac{1}{2}}$ waves of all wavelengths are possible for a given frequency ω, provided, of course, that they are all sufficiently short for this expression to be valid. The wavelength which is observed depends entirely on the nature of the oscillating disturbance.

Also in plate 1, figure 3, we see that the waves have been reflected at the side walls of the tank. Since a wave preserves its frequency, and since there is a one to one correspondence between frequency and inclination we see that the crest of the reflected wave must make the same angle with the horizontal as the incident wave. This is observed. However, we also have the condition that the component of wave-number parallel to the wall is conserved. If the incident and reflected waves make an angle θ with the horizontal and if the rigid wall makes an angle ϕ with the horizontal, it is easily shown that

$$|\mathbf{k}_i| \sin(\phi-\theta) = |\mathbf{k}_r| \sin(\phi+\theta),$$

or

$$\lambda_i^{-1} \sin(\phi-\theta) = \lambda_r^{-1} \sin(\phi+\theta),$$

where the suffices i and r refer to properties of the incident and reflected waves respectively. For a vertical wall, there is no change in the wavelength.

We also detect in plate 1 (3), fainter waves at larger angles to the horizontal, that is corresponding to a higher frequency. These waves correspond exactly to the first harmonics. We note that however closely these wave systems actually lie in space, they are essentially distinct. In no meaningful sense do waves proceed from one train to the other; their frequencies are different, their phase velocities are different and in general their wavelengths are different.

Plate 1 (4), shows waves corresponding to $\omega = 0{\cdot}419\omega_0$. The waves corresponding to the first harmonic are now approaching the vertical. In plate 1 (5), $\omega = 0{\cdot}615\omega_0$, so that the first harmonic is forbidden. This is clearly demonstrated.

In plate 1 (6), $\omega = 0{\cdot}699\omega_0$, and the angle is now sufficiently high for the small effects of inhomogeneity to be observed. As a result of having a linear rather than an exponential density distribution, ω_0 increases by about 5 % from its mean value in the upwards direction and decreases by about the same amount downwards. We would therefore expect the crests to be bent away from the vertical in the upper half plane and towards it in the lower half plane. This is shown in this plate and more markedly in plate 2 (7).

In plate 2 (7), which corresponds to $\omega = 0{\cdot}900\omega_0$, we see waves reflected from the upper and lower parts of the medium. In particular, this is not a simple reflexion from the free surface, but rather a trapping of the waves by the top layer of fluid. This arises as a result of the variation of density gradient near the free surface. There is an overall diffusion condition $d\rho/dy = 0$ at the free surface, so that although ω_0 rises slowly with increasing height from its mean value, by virtue of decreasing density, it decreases quite rapidly to zero in the uppermost layer of fluid. We have seen that double stationary points for short waves correspond to $k_2 = 0$ or $\omega = \omega_0$, and that they do not exist for waves of frequency ω less than ω_0 generated in a medium homogeneous in ω_0. However, if ω_0 changes and decreases below the value of ω pertaining to a particular wave train, on the curve in space along which $\omega_0 = \omega$ only double stationary points may exist. This curve, in the present case a horizontal line, presents a natural barrier to the wave train. Further, the ray theory suggests that the crests of the waves generated in a region of constant ω_0 greater than ω, will be bent as they enter a region of varying ω_0. If ω_0 decreases, they are bent towards the vertical and if ω_0 decreases to ω, they become vertical and have zero group velocity. The horizontal line along which $\omega_0 = \omega$ is therefore an envelope of cusps of the rays beyond which no small amplitude waves may propagate; the rays are reflected and describe a path which is the image of their incident path in the vertical axis through the cusp. In the neighbourhood of this trapping and reflexion the simple 'geometrical optics' approach of ray theory breaks down.

Precisely the same phenomenon occurs at a solid bottom where $d\rho/dy$ must also be zero, or at a sufficiently great depth in a fluid of constant density gradient, by virtue of increasing density. The first two effects can be seen in plate 2 (7).

Plate 2 (8), shows conditions for $\omega = 1{\cdot}11\omega_0$; no wave-like disturbances are observed in the steady state, disturbances being confined to a mixing region close to the cylinder. It will be observed in plates 1 and 2 (2–7), that the disturbances have a finite width of about one wavelength; this is close to the diameter of the bar.

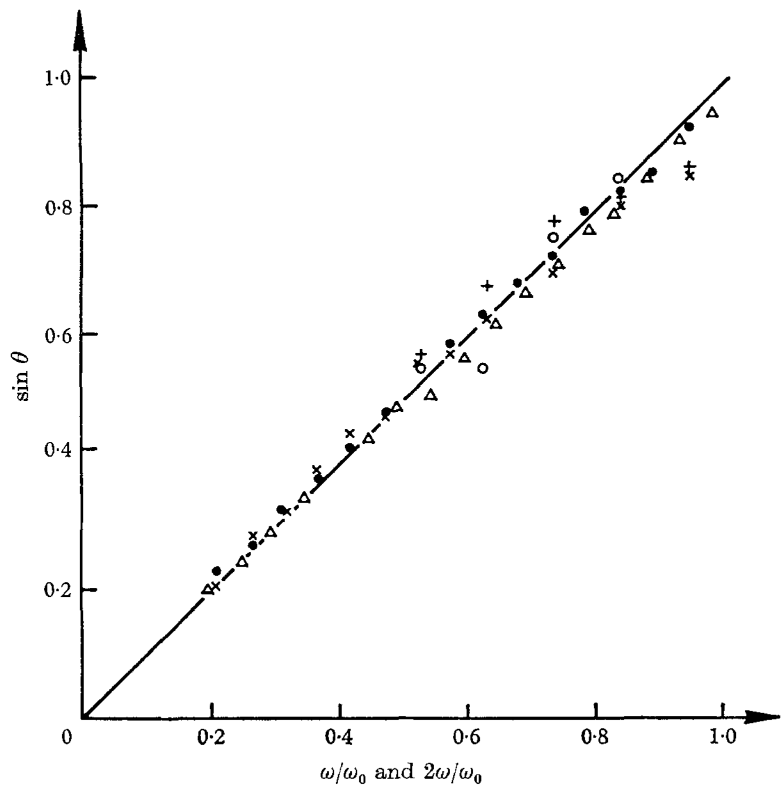

FIGURE 6. A graph of sin θ against ω/ω_0 and $2\omega/\omega_0$. × corresponds to $\omega_0 = 13{\cdot}25$ c/min; amplitude = 0·4 in. ● corresponds to $\omega_0 = 13{\cdot}25$ c/min; amplitude = 0·2 in. △ corresponds to $\omega_0 = 14{\cdot}1$ c/min; amplitude = 0·15 in. + corresponds to first harmonics of points marked ×. ○ corresponds to first harmonics of points marked ●.

The measurements taken from photographs of which plates 1 and 2 (1–8), are a small selection are summarized in figure 6. For fundamental frequencies sin θ is plotted against ω/ω_0 and for first harmonics, sin θ is plotted against $2\omega/\omega_0$. The Väisälä–Brunt frequency is taken to be constant and equal to its value at the mean level. The observed values fall below the line at the larger angles, where the effect of small inhomogeneity in ω_0 is noticeable. For these, the discrepancy, using the local value of ω_0, is less than 5 %.

Let us now compare the results of §4 with the schlieren photograph (plate 2 (9)) of the wave pattern produced by an impulsive disturbance. In this plate, the disturbing mechanism consisted of a flat strip 2 cm wide mounted horizontally with its sides vertical between struts of similar material. The strip was moved abruptly sideways through a distance of 1·5 cm and returned to its original position. Observation establishes that the number of crests does increase with time. The crests on plate 2 (9), appear not to emanate from the origin; this is due to distortion in the neighbourhood of the initial disturbance. The field of view corresponds to a small region near the origin in figure 3. The strip has excited the oblique waves; the cusps and transverse waves have not been detected.

We know from the work of Burnside (1888) and Havelock (1908) that the effect of the finite size of a disturbance is to reduce exponentially the amplitude of waves long compared with the disturbance, and to modulate more or less rapidly the amplitude of waves of wavelength comparable with the typical dimension of the disturbance, in the fashion of beats. This latter effect implies that near the disturbance, the crest and trough pattern may well appear to deviate from that produced by a point disturbance; group velocity, we recall, increases monotonically with wavelength. Further, one feels sure that short waves will be subjected to non-linear effects which will also change the pattern from that expected of the linear system from a point disturbance. Moreover, we would expect this non-linear region to expand uniformly in time, since the group velocity of waves of given wavelength is constant, the region being the circle or sphere marking the locus of those waves whose wavelength is the greatest to be affected by non-linearities for a given disturbance. We do not feel that we can say more than that the outer portion of the distorted region is due to the (linear) 'beats' phenomenon, but that it is not possible to determine whether any non-linear interactions have produced noticeable effects in the inner part of the distorted region.

6. Conclusion

The predictions of the small amplitude theory of the phase configurations of internal waves in a stratified fluid have been tested and confirmed.

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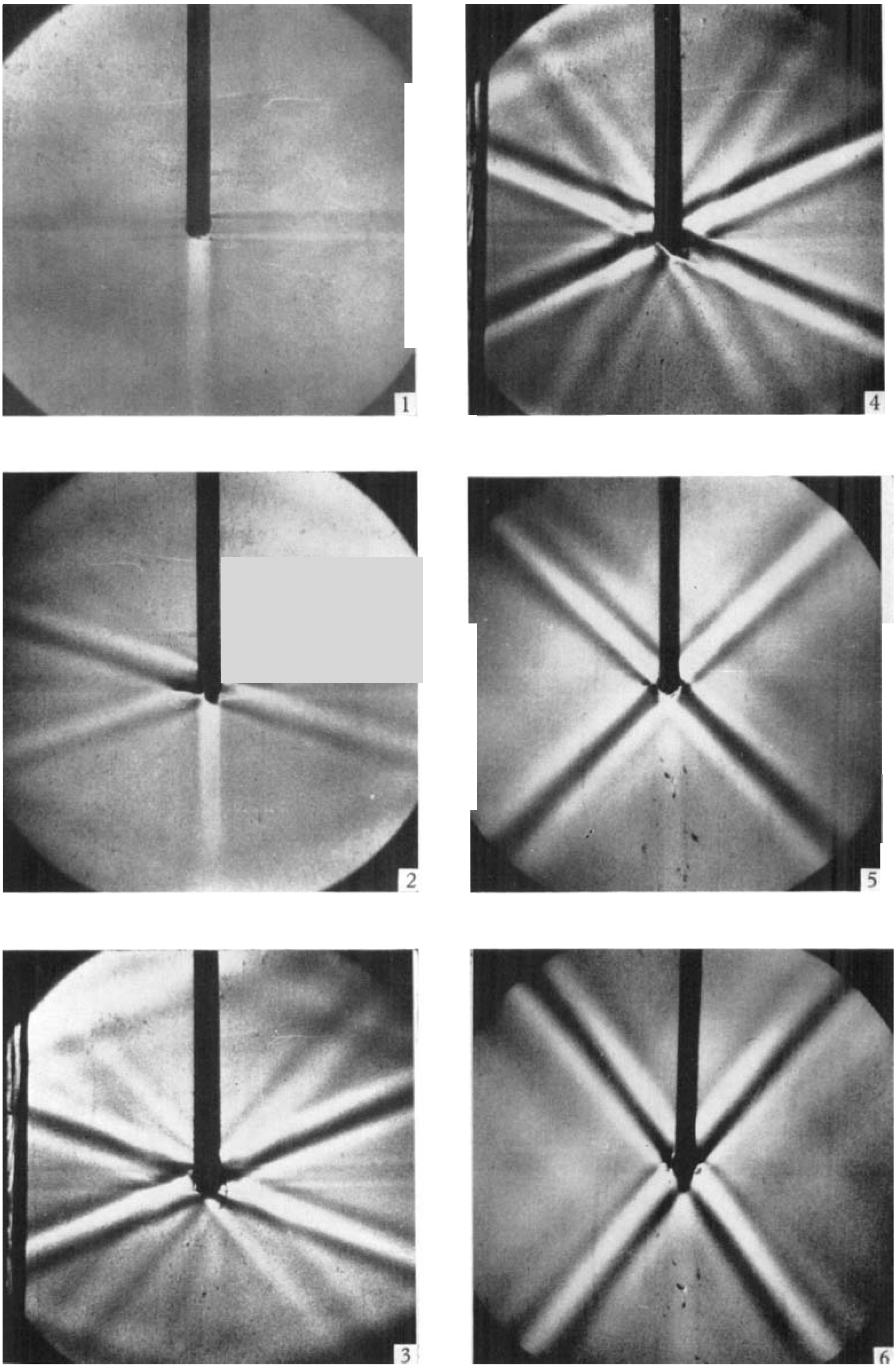

PLATE 1. (1) The image of the undisturbed fluid. (2) $\omega/\omega_0 = 0{\cdot}318$. (3) $\omega/\omega_0 = 0{\cdot}366$. (4) $\omega/\omega_0 = 0{\cdot}419$. (5) $\omega/\omega_0 = 0{\cdot}615$. (6) $\omega/\omega_0 = 0{\cdot}699$.

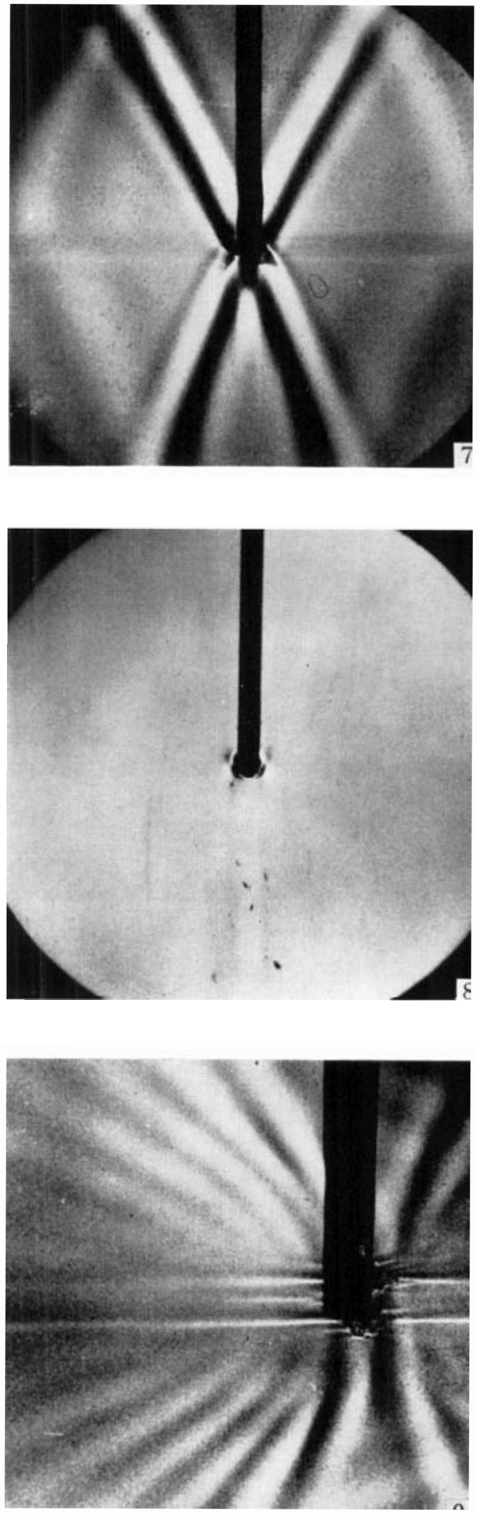

PLATE 2. (7) $\omega/\omega_0 = 0{\cdot}900$. (8) $\omega/\omega_0 = 1{\cdot}11$. (9) The wave system generated by an initial disturbance.